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On the Probability of Collision Between Climbing and Descending Aircraft

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The probability of coincidence (which is a upper bound for the probability of collision) is calculated for aircraft on arbitrary straight flight paths with constant speed: either aircraft may be climbing, descending, or in level flight and they may cross at any angle. Gaussian statistics with equal or distinct rms deviations for the two aircraft are used to calculate the probability of coincidence as a function of time in the first instance, with a correction factors for 1) the more accurate generalized error distribution; 2) asymmetry in vertical position errors. The time and distance of closest approach are used to calculate the position for maximum probability of coincidence. The cumulative probability of coincidence over all time is calculated, and expressed as a probability of coincidence per unit distance and per unit time: the latter is compared with the International Civil Aviation Organization target level of safety. Examples are given of the effect on coincidence probabilities of the seven parameters of the problem: velocities and glide slopes of each aircraft, crossing angle in horizontal projection, vertical separation, and rms position error.

Nomenclature

C	=	correction factor, Eq. (52)
F	=	generalized error distribution, Eq. (51)
f	=	correction factor used in Gaussian distribution, Eq. (A1)
H	=	altitude difference between the two flight levels, Eq. (2b)
P	=	Gaussian probability distribution, with zero mean, and rms deviation, Eq. (9)
$\bar{P}$	=	cumulative probability of coincidence over the time, Eq. (25)
$P^{(i)}$	=	probabilities of deviation of aircraft "i", Eq. (10)
$P^{(12)}$	=	probability of coincidence between aircraft "1" and "2", Eq. (11)
$P_*^{(12)}$	=	corrected probability of coincidence, Eq. (A4)
P_0	=	maximum probability of coincidence, at the position of most likely collision, Eq. (24)
$\bar{P}_*$	=	corrected cumulative probability of coincidence over the time, Eq. (A8)
$Q^{(i)}$	=	probability of coincidence of aircraft "i" per unit distance, Eq. (37)
R	=	International Civil Aviation Organization Target Level of Safety of 5×10^{-9}/h, Eq. (45)
$R^{(i)}$	=	probability of coincidence per unit time, Eq. (41)
$\mathbf{r}_i$	=	position vector of the aircraft "i", Eq. (2)
s	=	distance between the aircraft, Eq. (5)
$\Vert \mathbf{V} \Vert$	=	modulus of the vector $\mathbf{V}$, Eq. (5)
$\mathbf{V}_i$	=	velocity of the aircraft "i", Eq. (1)
$\mathbf{V}_1\mathbf{V}_2$	=	inner product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, Eq. (7a)
γ_i	=	glide slope of aircraft "i", Eq. (1)
Δz_i	=	vertical deviation of aircraft "i", Eq. (A1)
ε	=	asymmetry factor for vertical deviations, Eq. (A3)
λ_i	=	dimensionless velocity for aircraft "i", Eq. (40)
μ	=	dimensionless velocity factor, Eq. (35)
ν	=	joint asymmetric factor for vertical deviation of two aircraft, Eq. (A9)
σ_i	=	rms position error of aircraft "i", Eq. (9)
$\bar{\sigma}$	=	square root of arithmetic mean of variances, Eq. (B6b)
$\bar{\bar{\sigma}}$	=	geometric mean of rms position errors, Eq. (B6a)
χ	=	crossing angle, Eq. (1)

Subscripts

1	=	first aircraft
2	=	second aircraft

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I. Introduction

THE assessment of the effectiveness of separation standards [1] and collision avoidance measures [2] can be based [3] on safety metrics. An important class of safety metrics is specified by collision probabilities [4,5]. These are related to probabilities of coincidence [6], that is to say, the latter may be used as alternative safety metrics [7]. For the simplest case of two aircraft flying on parallel paths with equal constant velocity, the maximum probability of collision has been calculated [4], as well as cumulative probabilities of coincidence [6]. A distinct case is that of aircraft at different flight levels crossing at any angle: the maximum probability of coincidence can be calculated, as well as the cumulative probability of coincidence over time. The present paper considers the most general case of two aircraft flying at constant speed along two straight flight paths: either aircraft may be in level flight, climbing or descending, and they may cross at any angle and have any vertical separation.

The first preliminary step is to give a precise definition of vertical separation and crossing angle, in order to specify the relative position of the two aircraft (Sec. II). The latter is used, together with Gaussian statistics with identical rms position errors for the two aircraft, to calculate the probability of coincidence as a function of time (Sec. III). The time and distance of closest approach are then used to determine the maximum probability of coincidence (Sec. IV). Alternatively, the probability of coincidence can be integrated over all time, to specify a cumulative probability of coincidence (Sec. V); this can be expressed as a probability of coincidence per unit distance or per unit time (Sec. VI). The latter can be compared with the ICAO TLS standard (Sec. VII), whereas the former can be used to illustrate the effect of flight path geometry on coincidence probabilities (Sec. VIII). The Gaussian distribution gives the correct qualitative dependence on the probabilities of coincidence on the geometry of the encounter, but a more accurate quantitative estimate is provided by the generalized error distribution (Sec. IX). The introduction has

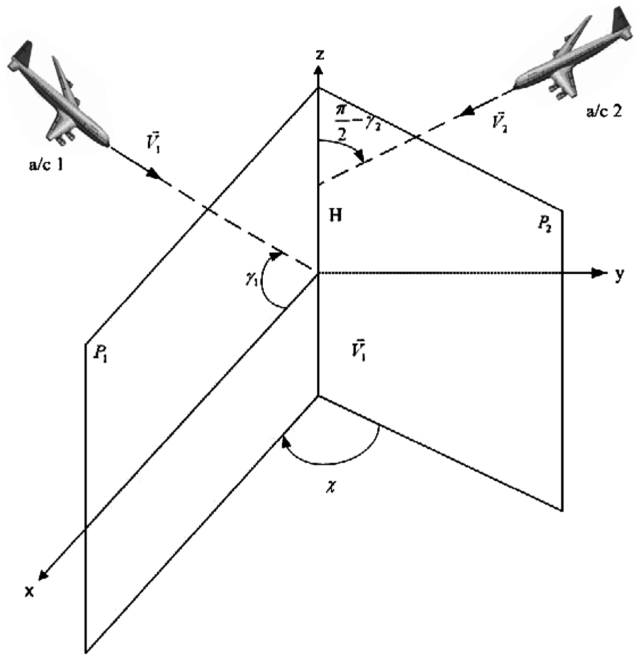

Fig. 1 Geometry of general crossing of two climbing and descending aircraft on straight flight paths.

outlined the motivation for the paper (Sec. I), and the Appendices introduce two extensions; namely the effects of: (Appendix A) asymmetry in vertical position errors; (Appendix B) distinct rms position errors for the two aircraft.

II. Generic Crossing of Two Straight Flight Paths

Consider two aircraft flying with velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ along straight flight paths, and let the vertical planes through the flight paths (Fig. 1) be respectively P_1 and P_2. The angle between these two planes is the crossing angle χ, as seen in an horizontal projection. The planes intersect along a vertical line, which is taken as the OZ axis, with origin on the path of the lower aircraft, which is by convention aircraft 1. The distance along the vertical to the flight path of the upper aircraft 2 is the vertical separation H. The XOZ plane is chosen as to contain the flight path of aircraft 1, so that its velocity has components:

$$\mathbf{V}_1 = V_1[\cos\gamma_1, 0, \sin\gamma_1] \tag{1a}$$

where γ_1 is the glide slope of aircraft 1. The velocity of aircraft 2 has components:

$$\mathbf{V}_2 = V_2[\cos\gamma_2\cos\chi, \cos\gamma_2\sin\chi, \sin\gamma_2] \tag{1b}$$

where γ_2 is the glide slope of aircraft 2. If the velocities are constant, the position vectors of the two aircraft are

$$\mathbf{r}_1 = \mathbf{V}_1 t \tag{2a}$$

$$\mathbf{r}_2 = H\mathbf{e_z} + \mathbf{V}_2 t \tag{2b}$$

and the Cartesian coordinates are

$$x_1 = V_1 t\cos\gamma_1 \tag{3a}$$

$$y_1 = 0 \tag{3b}$$

$$z_1 = V_1 t\sin\gamma_1 \tag{3c}$$

for the first aircraft, and

$$x_2 = V_2 t\cos\gamma_2\cos\chi \tag{4a}$$

$$y_2 = V_2 t\cos\gamma_2\sin\chi \tag{4b}$$

$$z_2 = H + V_2 t\sin\gamma_2 \tag{4c}$$

for the second aircraft.

The distance between the two aircraft is given by

$$s \equiv \|\mathbf{r}_1 - \mathbf{r}_2\| = |(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2|^{1/2} \tag{5}$$

either in terms of position vectors (2a) and (2b) or Cartesian coordinates (3a–3c) and (4a–4c). Using the former (2a) and (2b) in the distance (5):

$$s = \| -H\mathbf{e_z} + (\mathbf{V}_1 - \mathbf{V}_2)t\| \tag{6a}$$

it follows that

$$s^2 = H^2 + \|\mathbf{V}_2 - \mathbf{V}_1\|^2 t^2 + 2Ht\mathbf{e_z}\cdot(\mathbf{V}_2 - \mathbf{V}_1) \tag{6b}$$

where 1) the square of the modulus of the difference of velocities is given by

$$\|\mathbf{V}_1 - \mathbf{V}_2\|^2 = \|\mathbf{V}_1\|^2 + \|\mathbf{V}_2\|^2 - 2(\mathbf{V}_1\cdot\mathbf{V}_2) = (V_1)^2 + (V_2)^2 - 2V_1V_2(\cos\gamma_1\cos\gamma_2\cos\chi + \sin\gamma_1\sin\gamma_2) \tag{7a}$$

where (1a) and (1b) were used; 2) also using (1a) and (1b) the difference of vertical velocities is given by

$$(\mathbf{V}_2 - \mathbf{V}_1)\cdot\mathbf{e_z} = V_2\sin\gamma_2 - V_1\sin\gamma_1 \tag{7b}$$

Substitution of (7a) and (7b) in (6b) specifies the distance between the two aircraft as a function of time:

$$s^2 \equiv H^2 + 2Ht(V_2\sin\gamma_2 - V_1\sin\gamma_1) + [(V_1)^2 + (V_2)^2 - 2V_1V_2(\cos\gamma_1\cos\gamma_2\cos\chi + \sin\gamma_1\sin\gamma_2)]t^2 \tag{8}$$

The same result could be obtained by substituting (3a–3c) and (4a–4c) in (5), after some simplifications.

III. Probability of Coincidence as a Function of Time

For simplicity of calculation, it is assumed that Gaussian statistics specify the probability P of deviation r from flight path

$$P(r) = [1/(\sigma\sqrt{2\pi})]\exp[-r^2/(2\sigma^2)] \tag{9}$$

where σ is the rms position error; it will be found that this gives reliable qualitative results, and a quantitative correction factor will be introduced subsequently (Sec. IX). The probability distribution (9) does not take into account that flight path deviations in a vertical direction may be more probable downwards than upwards, due to the effect of gravity; this should became more marked close to the service ceiling of the aircraft. This aspect will be addressed in more detail in Appendix A. Concerning the Gaussian probability distribution (9), the case of dissimilar rms positions errors has been considered [6] for aircraft on the same horizontal parallel flight paths. The same method can be applied to the more general geometry considered here, that is to say, climbing and descending aircraft (see Appendix B): the simplification is made here that the two aircraft have negligibly different rms position errors. In this case the most likely position for a collision is halfway between the two aircraft (this is generalized in Appendix B).

Assuming the rms position error it is the same for both aircraft, the most likely position for collision is halfway between the two aircraft:

$$P^{(1)} = P^{(2)} = P(s/2) = [1/(\sigma\sqrt{2\pi})]\exp[-s^2/(8\sigma^2)] \tag{10}$$

Assuming that the flight path deviations of the two aircraft are statistically independent, the probability of coincidence is the product of probabilities

$$P^{(12)} = P^{(1)}P^{(2)} = [P(s/2)]^2 \tag{11}$$

and is given (10) by

$$P^{(12)} = [1/(2\pi\sigma^2)]\exp\{-[s/(2\sigma)]^2\} \tag{12}$$

as a function of the distance between the two aircraft. The relation between the probabilities of collision and coincidence is related to aircraft size, and is discussed elsewhere [4]; in particular cases, the aircraft size can be included either in the separation distance or in the rms position error [7]. The probability of coincidence can be used as an alternative safety metric to the collision probability [6].

Substituting (8) in (12), the probability of coincidence is given by

$$P^{(12)}(t) = [1/(2\pi\sigma^2)]\exp\{-[H/(2\sigma)]^2\} \times \exp\{[2H(\mathrm{V}_1 \sin\gamma_1 - \mathrm{V}_2 \sin\gamma_2)t]/(2\sigma)^2\}\exp\{-[(\mathrm{V}_1)^2 + (\mathrm{V}_2)^2 - 2\mathrm{V}_1\mathrm{V}_2(\cos\gamma_1\cos\gamma_2\cos\chi + \sin\gamma_1\sin\gamma_2)]t^2/(2\sigma)^2\} \tag{13}$$

as an explicit function of time. The first two factors specify the probability of coincidence at time $t = 0$, that is to say,

$$P^{(12)}(t) = [1/(2\pi\sigma^2)]\exp\{-[H/(2\sigma)]^2\}\exp(2bt - a^2t^2) \tag{14}$$

whereas the time dependence is specified by the two constants, that is to say,

$$2\sigma\mathrm{a} \equiv \|\mathbf{V_1} - \mathbf{V_2}\| = |(\mathrm{V}_1)^2 + (\mathrm{V}_2)^2 - 2\mathrm{V}_1\mathrm{V}_2(\cos\gamma_1\cos\gamma_2\cos\chi + \sin\gamma_1\sin\gamma_2)|^{1/2} \tag{15}$$

relates to the modulus of difference of velocities of the two aircraft, and

$$(2\sigma)^2 b \equiv H(\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2) = H(\mathbf{V_1} - \mathbf{V_2})\cdot\mathbf{e_z} \tag{16}$$

relates to the difference of vertical components of the velocities of the two aircraft.

IV. Conditions for Maximum Probability of Coincidence

In the case of two level flight paths $\gamma_1 = 0 = \gamma_2$, then $b = 0$ in (16) and it is clear that the smallest separation between the two aircraft is the vertical separation $s = H$, which occurs at time $t = 0$, corresponding to a probability of coincidence given by the first two factors of (14). For climbing and/or descending aircraft, the closest approach distance is not necessarily the vertical separation defined in Sec. II, and must be calculated. Note that the maximum probability of coincidence (12), corresponds to the minimum separation (8), where (15) and (16) can be used

$$S \equiv [s/(2\sigma)]^2 = [H/(2\sigma)]^2 - 2bt + a^2t^2 \tag{17}$$

It follows that

$$\mathrm{d}S/\mathrm{d}t = 2(a^2t - b) \tag{18a}$$

$$\mathrm{d}^2S/\mathrm{d}t^2 = 2a^2 > 0 \tag{18b}$$

so the extremum:

$$\mathrm{d}S/\mathrm{d}t = 0:\ t_m = b/a^2 \tag{19}$$

is a minimum (18b). Thus the time (19), (15), and (16)

$$t_m = H(\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2)/\|\mathbf{V}_1 - \mathbf{V}_2\|^2 \tag{20}$$

corresponds to the minimum separation and maximum probability of coincidence.

Substituting (20) in (2a) and (2b) it follows that the maximum probability of coincidence occurs when the aircraft are at positions

$$\mathbf{r}_{1,m} = (\mathbf{V}_1/\|\mathbf{V}_1 - \mathbf{V}_2\|^2)H(\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2) \tag{21a}$$

$$\mathbf{r}_{2,m} = H\mathbf{e_z} + (\mathbf{V}_2/\|\mathbf{V}_1 - \mathbf{V}_2\|^2)H(\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2) \tag{21b}$$

at a distance (17)

$$(s_m)^2 \equiv \|\mathbf{r}_{1,m} - \mathbf{r}_{2,m}\|^2 = H^2 + (2\sigma)^2[-2bt_m + (a^2t_m)^2] = H^2 - (2\sigma b/a)^2 = H^2\{1 - (\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2)^2/\|\mathbf{V}_1 - \mathbf{V}_2\|^2\} \tag{22}$$

the distance of closest approach (22) can be written, using (7a), in the form (23a):

$$s_m = H\sqrt{\mu} \tag{23a}$$

$$\mu\|\mathbf{V}_1 - \mathbf{V}_2\|^2 = (\mathrm{V}_1\cos\gamma_1)^2 + (\mathrm{V}_2\cos\gamma_2)^2 - 2\mathrm{V}_1\mathrm{V}_2\cos\gamma_1\cos\gamma_2\cos\chi \tag{23b}$$

where the parameter (23b) specifies whether it equals the vertical separation ($s_m = H$ for $\mu = 1$), or is larger ($s_m > H$ for $\mu > 1$) or is smaller ($s_m < H$ for $\mu < 1$). The maximum probability of coincidence is (12) thus

$$\mathrm{P}_0 = P^{(12)}(t_m) = [1/(2\pi\sigma^2)]\exp\{-[s_m/(2\sigma)]^2\} = [1/(2\pi\sigma^2)]\exp\{-\mu[H/(2\sigma)]^2\} \tag{24}$$

In the case of level crossing $\gamma_1 = 0 = \gamma_2$, then $\mu = 1$ in (23b), and the maximum probability of collision coincidence (24) is specified by the first two factors of (14).

V. Cumulative Probability of Coincidence over the Time

The cumulative probability of collision over time is defined by

$$\bar{\mathrm{P}} \equiv \int_{-\infty}^{+\infty} P^{(12)}(t)\,\mathrm{d}t \tag{25}$$

where (14) can be substituted:

$$\bar{P} = [1/(2\pi\sigma^2)]\exp\{-[H/(2\sigma)]^2\}\int_{-\infty}^{+\infty}\exp(2bt - a^2t^2)\,\mathrm{d}t \tag{26}$$

The latter integral can be related to the Gaussian integral

$$\int_{-\infty}^{+\infty}\exp(-\zeta^2)\,\mathrm{d}\zeta = \sqrt{\pi} \tag{27}$$

in the generalized form

$$\zeta \equiv at - \nu:\ \int_{-\infty}^{+\infty}\exp[-(at - \nu)^2]\,\mathrm{d}t = \sqrt{\pi}/a \tag{28}$$

which is obtained by a simple change of variable.

Putting the cumulative probability of coincidence over time (26) in the form

$$\bar{P} = [1/(2\pi\sigma^2)]\exp\{-[H/(2\sigma)]^2 + (b/a)^2\} \times \int_{-\infty}^{+\infty}\exp[-(at - b/a)^2]\,\mathrm{d}t \tag{29}$$

the value of the integral (28) is not affected by $\nu = b/a$:

$$\bar{P} = [1/(2\sigma^2 a\sqrt{\pi})]\exp\{(b/a)^2 - [H/(2\sigma)]^2\} \tag{30}$$

Using (15) and (16) it follows that

$$b/a \equiv [H/(2\sigma)](\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2)/\|\mathbf{V}_1 - \mathbf{V}_2\| \tag{31}$$

and substituting in (30) specifies

$$\bar{P} = [1/(\sigma\sqrt{\pi})](1/\|\mathbf{V}_1 - \mathbf{V}_2\|)\exp\{-[H/(2\sigma)]^2[1 - (\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2)^2/\|\mathbf{V}_1 - \mathbf{V}_2\|^2]\} \tag{32}$$

as the cumulative probability of coincidence over time.

VI. Probabilities of Coincidence per Unit Distance and per Unit Time

In the case of a level crossing, the cumulative probability of coincidence over time simplifies to

$$\gamma_1 = 0 = \gamma_2:\ \bar{P} = [1/(\sigma\sqrt{\pi})](1/\|\mathbf{V}_1 - \mathbf{V}_2\|)\exp\{-[H/(2\sigma)]^2\} \tag{33}$$

in the general case (32) of climbing or descending aircraft

$$\gamma_1 \neq 0 \neq \gamma_2:\ \bar{P} = [1/(\sigma\sqrt{\pi})](1/\|\mathbf{V}_1 - \mathbf{V}_2\|)\exp\{-\mu[H/(2\sigma)]^2\} \tag{34}$$

where the parameter μ in (23a) and (23b) was used:

$$\mu \equiv 1 - (\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2)^2/\|\mathbf{V}_1 - \mathbf{V}_2\|^2 \tag{35}$$

this parameter can also be expressed in terms of the a, b in (15) and (16):

$$\mu = 1 - [2\sigma b/(aH)]^2 \tag{36}$$

The cumulative probabilities of coincidence over time (33) and (34) become singular $\bar{P} = \infty$ for aircraft flying on parallel paths at the same speed $\mathbf{V}_1 = \mathbf{V}_2$; in this case the cumulative probabilities of coincidence integrated over position should be used instead [6].

The probability of coincidence as a function of time (13) in general, and the maximum probability of coincidence (24) and (23b) in particular, have the dimensions of inverse square of distance. The cumulative probability of coincidence integrated over time (32) has the dimensions of time divided by the square of distance. Multiplied by a velocity, it becomes a probability of coincidence per unit distance:

$$Q^{(1)} = \bar{P}V_1 \tag{37a}$$

$$Q^{(2)} = \bar{P}V_2 \tag{37b}$$

respectively, for the first (37a) and second (37b) aircraft; substitution of (34) in (37a) and (37b) leads to

$$Q^{(1)}, Q^{(2)} = \{\lambda_1, \lambda_2\}Q_0 \tag{38}$$

where

$$Q_0 \equiv [1/(\sigma\sqrt{2\pi})]\exp\{-\mu[H/(2\sigma)]^2\} \tag{39}$$

and the factors

$$\{\lambda_1, \lambda_2\} = \sqrt{2}\{V_1, V_2\}\|\mathbf{V}_1 - \mathbf{V}_2\|^{-1} \tag{40}$$

become unity for aircraft with identical velocities $V_1 = V_2$ crossing level $\gamma_1 = 0 = \gamma_2$ at right angles $\chi = \pi/2$, because $\|\mathbf{V}_1 - \mathbf{V}_2\| = \sqrt{2}V_1 = \sqrt{2}V_2$.

VII. Effects of Separation, Velocities, and Errors

To obtain a probability of coincidence per unit time, it is necessary to multiply by the velocity once more

$$R^{(1)} = Q^{(1)}V_1, \qquad R^{(2)} = Q^{(2)}V_2 \tag{41}$$

or, using (37a) and (37b)

$$R^{(1)} = \bar{P}(V_1)^2, \qquad R^{(2)} = \bar{P}(V_2)^2 \tag{42}$$

alternatively (34)

$$R^{(1)}, R^{(2)} = [1/(\sigma\sqrt{\pi})]\{(V_1)^2, (V_2)^2\}\|\mathbf{V}_1 - \mathbf{V}_2\|^{-1}\exp\{-\mu[H/(2\sigma)]^2\} \tag{43}$$

where (39) can be used

$$R^{(1)}, R^{(2)} = \sqrt{2}\{(V_1)^2, (V_2)^2\}\|\mathbf{V}_1 - \mathbf{V}_2\|^{-1}Q_0 \tag{44}$$

as alternate expression for the probabilities of coincidence per unit time. The latter can be compared with the ICAO Target Level of Safety (TLS):

$$R^{(1)}, R^{(2)} \leq R = 5 \times 10^{-9} \text{ per hour} \tag{45}$$

The probabilities of coincidence (44) and (39) depend on seven parameters:

$$R^{(1)}, R^{(2)} = f(\sigma; H, \chi; V_1, V_2; \gamma_1, \gamma_2) \tag{46}$$

namely: 1) the rms position error σ, which is assumed to be the same for both aircraft; (ii/iii) the vertical separation H and crossing angle χ, as defined in Sec. II; (iv/v) the airspeeds V_1, V_2 of the two aircraft; (vi/vii) the flight path or glide path angles γ_1, γ_2. As an example consider one aircraft descending $\gamma_1 = -3$ deg at $V_1 = 500$ kt, and another climbing $\gamma_2 = 30$ deg at $V_1 = 300$ kt, and crossing at an angle $\chi = 60$ deg. The modulus of the difference of velocities (7a) is $\|\mathbf{V}_1 - \mathbf{V}_2\| = 467$ kt; the difference of vertical velocities $\mathrm{V}_1\sin\gamma_1 - \mathrm{V}_2\sin\gamma_2 = -176$ kt leads to the value $\mu = 0.858$ for the parameter (35). The vertical separation $H = 1000$ ft and rms position error $\sigma = 80$ ft, lead to $\exp[-\mu H^2/(4\sigma^2)] = 2.78 \times 10^{-15}$, and $1/(\sigma\sqrt{2\pi}) = 30.3$ per nautical mile, so the product of these two factors (39) is $Q_0 = 8.42 \times 10^{-14}$ per nm. The remaining factor in (44) leads to the probabilities of coincidence per unit time $R^{(1)} = 6.37 \times 10^{-11}$ and $R^{(2)} = 2.29 \times 10^{-11}$ per hour. Both comply with the ICAO TLS Standard (45), so the rms position error $\sigma = 80$ ft is safe. A rms position error $\sigma = 90$ ft would lead to $R^{(1)} = 6.45 \times 10^{-8}$ and $R^{(2)} = 2.32 \times 10^{-8}$ per hour, and thus no longer comply with the ICAO TLS standard (45) for either aircraft.

VIII. Effect of Glide Slopes and Crossing Angle

The probability of coincidence per unit distance (38) depends only on six parameters

$$Q^{(1)}, Q^{(2)} = g(\sigma; H, \chi; V_2/V_1; \gamma_1, \gamma_2) \tag{47}$$

instead of seven in the probability of coincidence per unit time (46), because the velocities appear only as a ratio in (40) and (7a):

$$\lambda_1 = \sqrt{2}|1 + (V_2/V_1)^2 - 2(V_2/V_1)(\cos\gamma_1\cos\gamma_2\cos\chi + \sin\gamma_1\sin\gamma_2)|^{-1/2} \tag{48a}$$

$$\lambda_2 = \text{interchange}(V_1, V_2) \tag{48b}$$

and also in the parameter μ in (35):

$$1 - \mu = \{\lambda_1[\sin\gamma_1 - (V_2/V_1)\sin\gamma_2]\}^2/2 \tag{49}$$

Since the probabilities of coincidence are interchanged exchanging V_1, V_2:

$$Q^{(1)}(\sigma; H, \chi; V_2/V_1; \gamma_1, \gamma_2) = Q^{(2)}(\sigma; H, \chi; V_1/V_2; \gamma_1, \gamma_2) \tag{50}$$

and γ_1, γ_2, it is sufficient to calculate $Q^{(1)}$ in the examples that follow. Both examples concern a vertical separation $H = 1000$ ft and a rms position error $\sigma = 90$ ft. In the first example (Table 1) the aircraft cross at right angles $\chi = 90$ deg at speeds $V_1 = 500$ kt and $V_2 = 400$ kt, and the effect of level flight $\gamma_1, \gamma_2 = 0$ deg, and climb $\gamma_1, \gamma_2 = 3$ deg, 30 deg, or descent $\gamma_1, \gamma_2 = -3$ deg, -30 deg are shown. Taking as reference the level crossing $\gamma_1 = 0$ deg $=\gamma_2$, it is seen that: 1) if the lower aircraft descends $\gamma_1 < 0$ deg the probability of coincidence is less, because it moves farther from the upper aircraft; 2) if the lower aircraft climbs $\gamma_1 > 0$ deg the probability of coincidence increases, because it moves closer to the upper aircraft; 3) if the upper aircraft descends $\gamma_2 < 0$ deg the probability of coincidence increases, because it comes closer to the lower aircraft: (iv) if the upper aircraft climbs, $\gamma_2 > 0$ deg, the probability of coincidence decreases, because it moves away from the lower aircraft. There are exceptions to these general trends, for example, if one aircraft flies level $\gamma_1 = 0$, then the climb or descent $\pm\gamma_2$ of the other aircraft gives the same probabilities of coincidence, as follows from Eqs. (38), (39), (48a), and (48b). The remaining values in Table 1 show probabilities of coincidence increasing when the aircraft approach further, and decreasing when they move away from each other. Similar conclusions are drawn from Table 2, where the two aircraft have the same speed $V_1 = 500$ kt $= V_2$, leading to lower probabilities of coincidence. The aircraft speeds are kept identical in Table 3 where the crossing angle changes to $\chi = \pi/6$, leading to higher probabilities of coincidence. The probabilities of coincidence are minimum for aircraft flying on opposite direction $\chi = \pi$ as seen on Table 4.

Table 1 Probabilities of coincidence per nautical mile, for aircraft with vertical separation $H = 1000$ ft, rms position error $\sigma = 90$ ft, crossing at right angles $\chi = 90$ deg, at velocities $V_1 = 500$ and $V_2 = 400$ kt. Effect of five climb or descent angles γ_1 for the first aircraft and γ_2 for the second aircraft

$Q^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	5.27×10^{-13}	3.74×10^{-12}	7.26×10^{-12}	1.51×10^{-11}	6.73×10^{-08}
−3	2.09×10^{-11}	3.59×10^{-13}	3.69×10^{-13}	4.22×10^{-13}	7.72×10^{-11}
0	3.95×10^{-11}	3.76×10^{-13}	3.57×10^{-13}	3.76×10^{-13}	3.95×10^{-11}
3	7.72×10^{-11}	4.22×10^{-13}	3.69×10^{-13}	3.59×10^{-13}	2.09×10^{-11}
30	6.73×10^{-08}	1.51×10^{-11}	7.26×10^{-12}	3.74×10^{-12}	5.27×10^{-13}
$R^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	2.64×10^{-10}	1.87×10^{-09}	3.63×10^{-09}	7.54×10^{-09}	3.37×10^{-05}
−3	1.04×10^{-08}	1.79×10^{-10}	1.85×10^{-10}	2.11×10^{-10}	3.86×10^{-08}
0	1.97×10^{-08}	1.88×10^{-10}	1.79×10^{-10}	1.88×10^{-10}	1.97×10^{-08}
3	3.86×10^{-08}	2.11×10^{-10}	1.85×10^{-10}	1.79×10^{-10}	1.04×10^{-08}
30	3.37×10^{-05}	7.54×10^{-09}	3.63×10^{-09}	1.87×10^{-09}	2.64×10^{-10}

Table 2 As Table 1 for aircraft with the same velocities $V_1 = V_2 = 500$ kt

$Q^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	3.74×10^{-13}	7.85×10^{-12}	1.53×10^{-11}	3.14×10^{-11}	6.66×10^{-08}
−3	7.85×10^{-12}	3.24×10^{-13}	3.38×10^{-13}	3.83×10^{-13}	3.14×10^{-11}
0	1.53×10^{-11}	3.38×10^{-13}	3.24×10^{-13}	3.38×10^{-13}	1.53×10^{-11}
3	3.14×10^{-11}	3.83×10^{-13}	3.38×10^{-13}	3.24×10^{-13}	7.85×10^{-12}
30	6.66×10^{-08}	3.14×10^{-11}	1.53×10^{-11}	7.85×10^{-12}	3.74×10^{-13}
$R^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	1.87×10^{-10}	3.93×10^{-09}	7.67×10^{-09}	1.57×10^{-08}	3.33×10^{-05}
−3	3.93×10^{-09}	1.62×10^{-10}	1.69×10^{-10}	1.91×10^{-10}	1.57×10^{-08}
0	7.67×10^{-09}	1.69×10^{-10}	1.62×10^{-10}	1.69×10^{-10}	7.67×10^{-09}
3	1.57×10^{-08}	1.91×10^{-10}	1.69×10^{-10}	1.62×10^{-10}	3.93×10^{-08}
30	3.33×10^{-05}	1.57×10^{-08}	7.67×10^{-09}	3.93×10^{-09}	1.87×10^{-10}

Table 3 As Table 2 with an acute crossing angle $\chi = \pi/6$

$Q^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	1.02×10^{-12}	6.42×10^{-07}	3.26×10^{-06}	1.46×10^{-05}	6.05×10^{-02}
−3	6.42×10^{-07}	8.86×10^{-13}	1.20×10^{-12}	2.93×10^{-12}	1.46×10^{-05}
0	3.26×10^{-06}	1.20×10^{-12}	8.84×10^{-13}	1.20×10^{-12}	3.26×10^{-06}
3	1.46×10^{-05}	2.93×10^{-12}	1.20×10^{-12}	8.86×10^{-13}	6.42×10^{-07}
30	6.05×10^{-02}	1.46×10^{-05}	3.26×10^{-06}	6.42×10^{-07}	1.02×10^{-12}
$R^{(1)}$			γ_1 (deg)		
γ_2 (deg)	−30	−3	0	3	30
−30	5.11×10^{-10}	3.21×10^{-04}	1.63×10^{-03}	7.31×10^{-03}	$3.03 \times 10^{+01}$
−3	3.21×10^{-04}	4.43×10^{-10}	6.02×10^{-10}	1.46×10^{-09}	7.31×10^{-03}
0	1.63×10^{-03}	6.02×10^{-10}	4.42×10^{-10}	6.02×10^{-10}	1.63×10^{-03}
3	7.31×10^{-03}	1.46×10^{-09}	6.02×10^{-10}	4.43×10^{-10}	3.21×10^{-04}
30	$3.03 \times 10^{+01}$	7.31×10^{-03}	1.63×10^{-03}	3.21×10^{-04}	5.11×10^{-10}

Table 4 As Table 2 for aircraft flying in opposite direction $\chi = \pi$

$Q^{(1)}$	γ_1 (deg)				
γ_2 (deg)	−30	−3	0	3	30
−30	2.64×10^{-13}	1.28×10^{-12}	1.87×10^{-12}	2.84×10^{-12}	5.14×10^{-10}
−3	1.28×10^{-12}	2.29×10^{-13}	2.34×10^{-13}	2.49×10^{-13}	2.84×10^{-12}
0	1.87×10^{-12}	2.34×10^{-13}	2.29×10^{-13}	2.34×10^{-13}	1.87×10^{-12}
3	2.84×10^{-12}	2.49×10^{-13}	2.34×10^{-13}	2.29×10^{-13}	1.28×10^{-12}
30	5.14×10^{-10}	2.84×10^{-12}	1.87×10^{-12}	1.28×10^{-12}	2.64×10^{-13}
$R^{(1)}$	γ_1 (deg)				
γ_2 (deg)	−30	−3	0	3	30
−30	1.32×10^{-10}	6.42×10^{-10}	9.37×10^{-10}	1.42×10^{-09}	2.57×10^{-07}
−3	6.42×10^{-10}	1.15×10^{-10}	1.17×10^{-10}	1.25×10^{-10}	1.42×10^{-09}
0	9.37×10^{-10}	1.17×10^{-10}	1.14×10^{-10}	1.17×10^{-10}	9.37×10^{-10}
3	1.42×10^{-09}	1.25×10^{-10}	1.17×10^{-10}	1.15×10^{-10}	6.42×10^{-10}
30	2.57×10^{-07}	1.42×10^{-09}	9.37×10^{-10}	6.42×10^{-10}	1.32×10^{-10}

IX. Effect of Choice of Probability Distribution

The qualitative results, on the dependence of the probabilities of coincidence, on the encounter geometry, are independent of the probability distribution assumed; in this respect Gaussian probabilities are useful in allowing a full analytical development in Sec. II, III, IV, V, VI, VII, and VIII. On the other hand, Gaussian probabilities are known to under estimate [4] the probabilities of collision, because they are suited to frequent events [8], whereas collisions are very rare events [9]. The generalized error distribution [10], provides a good fit to observed aircraft flight path deviations in bi-modal form [5], and remains reasonably accurate in the simpler [10] unimodal form

$$F(x) = (\sqrt{15/2}/\sigma)\exp\{-\sqrt[4]{120}\sqrt{|x|/\sigma}\} \tag{51}$$

The replacement of Gaussian (9) by generalized error distribution (51), corresponds to a correction factor

$$C \geq |F(x)/P(x)|^2 = 15\pi\exp\{(x/\sigma)^2 - 2\sqrt[4]{120}\sqrt{|x|/\sigma}\} \equiv G(x) \tag{52}$$

because the coincidence probabilities are quadratic.

The correction factor is evaluated at the point of most likely collision, i.e., at half the minimum separation distance (23a):

$$C(\mu) = G(x = H\sqrt{\mu}/2) = 15\pi\exp\{\mu[H/(2\sigma)]^2 - \sqrt[4]{120\mu}\sqrt{2H/\sigma}\} \tag{53}$$

A simpler estimate of (53), which depends only on altitude separation, is to set $\mu = 1$:

$$\bar{C} \equiv G(x = H/2) = 15\pi\exp\{[H/(2\sigma)]^2 - \sqrt[4]{120}\sqrt{2H/\sigma}\} \tag{54}$$

In Table 1–4 the vertical separation is $H = 1000$ ft and the rms position error $\sigma = 90$ ft, and thus the correction factor (54) is $\bar{C} = 2.00 \times 10^8$. The generalized error distribution thus leads to much higher probabilities of coincidence than the Gaussian. In some instances the probabilities of coincidence exceed unity, showing that the correction factor (54) is too crude, and should be refined although, as it stands, it errs on the safe side. Since the correction factor is constant the qualitative conclusions are unchanged.

X. Discussion

The discussion concerns the types of probabilities to be used for the various encounter geometries and the relevant notations in nomenclature. The method applies to any probability distribution P for deviations from the flight path, which can be applied to the first $P^{(1)}$ and second $P^{(2)}$ aircraft to specify the probability of coincidence $P^{(12)}$. The maximum of the latter P_0, at the position or time of most likely coincidence, can be interpreted as a cumulative probability of coincidence in a zero dimensions, i.e., at a point. By integrating over space, it is possible to obtain cumulative probabilities of coincidence in one P_1, two P_2 or three P_3 dimensions. And integration over time leads to the cumulative probability of coincidence over time $\bar{P}$.

The probability of collision has for upper bound the probability of coincidence [7]; the latter and its maximum value, has the dimensions of inverse square of distance $P^{(12)} \sim l^{-2} \sim P_0$; the cumulative probability of collision over time multiplies this by time $\bar{P} \sim l^{-2}\tau$. The cumulative probabilities of collision in space is one dimension scale on inverse distance $P_1 \sim l^{-1}$, in three dimensions scale on distance $P_3 \sim l$, and in two dimensions are dimensionless, i. e. P_2 has no units. The velocity V may be used to specify a probability of collision per unit length $Q \sim l^{-1}$ or per unit time $R \sim \tau^{-1}$, in the case of aircraft flying at the same speed, i.e., on parallel tracks. For more general encounter geometries, i.e., aircraft flying at different speeds V_1, V_2 or directions, there are two probabilities of collision per unit distance $Q^{(1)} \sim l^{-1} \sim Q^{(2)}$, and per unit time $R^{(1)} \sim \tau^{-1} \sim R^{(2)}$. The probabilities of collision per unit time can be compared with the International Civil Aviation Organization Target Level of Safety (ICAO TLS) Standard $R = 5 \times 10^{-9}$ per hour.

The calculation of collision probabilities is based on the penetration by another aircraft of a volume enclosing the reference aircraft [4]. This method has been extended by relaxing some of the assumptions [11], and has been applied not only to cruise flight but also to the landing phase [12]. There are other methods of assessing collision probabilities at landing [13]. The probabilities of collision for individual aircraft pairs are the building blocks to assess the overall collision probability of air traffic control scenario or sector [4,14]. Thus it is useful to have the simplest approach to collision probabilities for an aircraft pair; such an approach is the use of the probability of coincidence as an upper bound to probabilities of collision [7], which has been applied to aircraft on parallel tracks [6], and is extended here to crossing trajectories.

Appendix A: Effect of Asymmetry of Vertical Position Errors

The Gaussian probability distribution (9) is symmetric in the deviation r, and thus is more suitable for horizontal deviations, which are not affected by gravity, than for vertical deviations. For the latter, the effect of the weight will be to make downward deviations (i.e., descents) more likely than upward deviations (i.e., ascents); this asymmetry of vertical deviations will be more marked closer to the service ceiling, and at lower altitudes, for steeper climb angles. The generalized error distribution (51) was developed to model vertical deviations of horizontal flight paths [5,10]. One way to render explicit the effect of the weight on the vertical asymmetry of climbing or descending flight paths, is to modify the Gaussian (9) with a correction factor:

$$P_*(r) = [1 - f(\Delta z)]P(r) \tag{A1}$$

where f is an odd monotonic increasing function of the vertical deviation:

$$f'(\Delta z) > 0 \tag{A2a}$$

$$f(-|\Delta z|) = -f(|\Delta z|) \tag{A2b}$$

so that 1) horizontal deviations $\Delta z = 0$ do not change $f(0) = -f(-0) = 0$ the probability $P_*(r) = P(r)$; 2) upward deviations $\Delta z > 0$ reduce the probability $P_*(r) < P(r)$ because $f(\Delta z > 0) > 0$ since f is a monotonic increasing function; 3) downward deviations $\Delta z < 0$ increase the probability $P_*(r) > P(r)$ for the same reason $f(\Delta z < 0) < 0$ that the function is monotonic increasing. The simplest example of monotonic increasing function is the linear function:

$$\varepsilon > 0\colon\ f(\Delta z) = \varepsilon \Delta z \tag{A3}$$

where the coefficient ε is constant and positive.

Introducing the factor (A1) in (11) modifies the probability of coincidence to

$$P_*^{(12)} = [1 - f(\Delta z_1)][1 - f(\Delta z_2)]P^{(12)} \tag{A4}$$

where $P^{(12)}$ is still given by (12), and Δz_1, Δz_2 are the vertical deviations of, respectively, the first and second aircraft, and may be different. Assuming that the probabilities of deviation are small but not negligible, in the sense $f^2 \ll 1$, then (A4) simplifies to

$$f^2 \ll 1\colon\ P_*^{(12)} = [1 - f(\Delta z_1) - f(\Delta z_2)]P^{(12)} \tag{A5}$$

The vertical deviations of the two aircraft should be proportional to their vertical velocity and to time:

$$\Delta z_1 = \mathrm{V}_1 \sin(\gamma_1)t, \qquad \Delta z_2 = \mathrm{V}_2 \sin(\gamma_2)t \tag{A6}$$

and in the case (A3), lead by substitution in (A5) to

$$f^2 \ll 1\colon\ P_*^{(12)} = \{1 - [\varepsilon_1 \mathrm{V}_1 \sin(\gamma_1) + \varepsilon_2 \mathrm{V}_2 \sin(\gamma_2)]t\}P_*^{(12)} \tag{A7}$$

where the factor ε may be different for the two aircraft, that is, one may have larger vertical deviations than the other, even for similar vertical velocities. The factor (A7) now appears in the cumulative probability of collision (25), which becomes

$$\bar{P}_* = \int_{-\infty}^{+\infty} (1 - \nu t)P^{(12)}(t)\,\mathrm{d}t \tag{A8}$$

where

$$\nu \equiv \varepsilon_1 V_1 \sin(\gamma_1) + \varepsilon_2 \mathrm{V}_2 \sin(\gamma_2) \tag{A9}$$

includes the effects of the vertical deviations of both aircraft.

The integral (A8) may be evaluated like (26), that is to say,

$$\bar{P}_*(2\pi\sigma^2)\exp\{[H/(2\sigma)]^2\} = \int_{-\infty}^{+\infty} [1 - \nu t]\exp(2bt - at^2)\,\mathrm{d}t \tag{A10}$$

except that there is an extra factor in square brackets in the integrand. Nothing that differentiation of the exponential with regard to b is equivalent to multiplication by $2t$:

$$\bar{P}_*(2\pi\sigma^2)\exp\{[H/(2\sigma)]^2\} = [1 - (\nu/2)(\partial/\partial b)]\int_{-\infty}^{+\infty} \exp(2bt - at^2)\,\mathrm{d}t \tag{A11}$$

allows putting the factor in square brackets outside the integral; the integral is now the same as in (26), and can be evaluated similarly (27–29) as

$$\begin{aligned}\bar{P}_* &= [1/(2\pi\sigma^2)]\exp\{-[H/(2\sigma)]^2\}[1 - (\nu/2)(\partial/\partial b)]\exp(b^2/a^2)\\ &= [1/(2\pi\sigma^2)]\exp[b^2/a^2 - [H/(2\sigma)]^2](1 - \nu b/a^2)\\ &= \bar{P}(1 - \nu b/a^2)\end{aligned} \tag{A12}$$

Thus the cumulative probability of collision over time without $\bar{P}$ and with $\bar{P}_*$ vertical deviations, are related as a relative change by

$$1 - \bar{P}_*/\bar{P} = \nu b/a^2 = \nu H[(\mathbf{V_1} - \mathbf{V_2}) \cdot \mathbf{e_z}]\|\mathbf{V_1} - \mathbf{V_2}\|^{-2} \tag{A13}$$

where (15) and (16) were used. The effect of vertical deviations is negligible if (A13) is very small compared with unity, and nonnegligible otherwise. For example, for aircraft with velocities $V_1 = 500$ and $V_2 = 400$ kt and glide slopes $\gamma_1 = +30$ and $\gamma_2 = -30$ deg, the difference of vertical velocities is $[(\mathbf{V_1} - \mathbf{V_2}) \cdot \mathbf{e_z}] = V_1 \sin(\gamma_1) - \mathrm{V}_2 \sin(\gamma_2) = 450$ kt; if the aircraft cross orthogonally $\chi = 90$ deg, the square of the velocity difference (7a) is

$$\mathbf{V_1} \cdot \mathbf{V_2} = 0\colon\ \|\mathbf{V_1} - \mathbf{V_2}\| = (V_1)^2 + (V_2)^2 - 2V_1V_2\cos(\gamma_1 - \gamma_2) \tag{A14}$$

and gives $\|\mathbf{V_1} - \mathbf{V_2}\|^2 = 2.1 \times 10^5$ kt^2. Using a vertical separation $H = 1000$ ft $= 0.165$ nm, and (A9) a vertical deviation $\nu = \varepsilon_1 \times 50$ kt for $\varepsilon_1 = \varepsilon_2 = 0.1$/nm, leads to $\nu = 5$ kt/nm; hence the factor in (A13) becomes $5 \times 0.165 \times 450/(2.1 \times 10^5) = 1.76 \times 10^{-3}$ which is quite small compared with unity. In this particular case the asymmetry of altitude deviations due to weight was not important, but it could be in other situations, depending on the value of (A13).

Appendix B: Effect of Dissimilar Position Errors for the Two Aircraft

In the case where the two aircraft have dissimilar rms position errors $\sigma_1 \neq \sigma_2$, the Gaussian statistics (9) are distinct for the first:

$$P_1(r) = [1/(\sigma_1\sqrt{2\pi})]\exp[-r^2/(2\sigma_1^2)] \tag{B1a}$$

and second

$$P_2(r) = [1/(\sigma_2\sqrt{2\pi})]\exp[-r^2/(2\sigma_2^2)] \tag{B1b}$$

aircraft. The most likely point of collision is a weighted mean [6] of the closest separation:

$$r_1 = \mathrm{s}/[1 + (\sigma_2/\sigma_1)^2] \tag{B2a}$$

this reduces to $r_1 = \mathrm{s}/2$ halfway in the case of equal rms position errors $\sigma_2 = \sigma_1$, but $r_1 \neq r_2$ otherwise

$$r_2 = s - r_1 = \mathrm{s}/[1 + (\sigma_1/\sigma_2)^2] \tag{B2b}$$

Thus the probability of coincidence (11) is now given by

$$P^{(12)} = P_1(r_1)P_2(r_2) \tag{B3}$$

and using (B1a), (B1b), (B2a), and (B2b) leads to

$$P^{(12)} = [1/(2\pi\sigma_1\sigma_2)]\exp\{-s^2/[2(\sigma_1^2 + \sigma_2^2)]\} \tag{B4}$$

In the case of equal $\sigma_2 = \sigma_1 = \sigma$ rms position errors for the two aircraft (B4) simplifies to (12).

In the general case of dissimilar rms position errors for the two aircraft $\sigma_1 \neq \sigma_2$, it is possible to write (B4) in the form similar to (12), that is to say,

$$P^{(12)} = [1/(2\pi\bar{\bar{\sigma}}^2)]\exp[-s^2/(2\bar{\sigma})^2] \tag{B5}$$

where

$$\bar{\bar{\sigma}} \equiv \sqrt{\sigma_1\sigma_2} \tag{B6a}$$

$$\bar{\sigma}^2 \equiv (\sigma_1^2 + \sigma_2^2)/2 \tag{B6b}$$

are introduced: 1) the geometric mean of the rms position errors (B6a); 2) the arithmetic mean of the variances (B6b). For example, if one rms position error is twice the other $\sigma_2 = 2\sigma_1$, the values (B6a) and (B6b) are

$$\sigma_2 = 2\sigma_1 \equiv 2\sigma\colon \bar{\bar{\sigma}} = \sigma\sqrt{2}, \qquad \bar{\sigma} = \sigma\sqrt{5/2} \tag{B7}$$

In the general case of distinct rms position errors $\sigma_2 \neq \sigma_1$, it is necessary to distinguish between (B6a) and (B6b), for example, it is $\bar{\sigma}$ but not $\bar{\bar{\sigma}}$ which appears in (15) and (16), that is to say,

$$2\bar{\sigma}a = \|\mathbf{V_1} - \mathbf{V_2}\| \tag{B8a}$$

$$(2\bar{\sigma})^2 b = H(\mathbf{V_1} - \mathbf{V_2}) \cdot \mathbf{e_z} \tag{B8b}$$

In the integration (25) for the cumulative probability of collision over time, (26) is substituted by

$$\bar{P} = [1/(2\pi\bar{\bar{\sigma}}^2)] \exp\{-[H/(2\bar{\sigma})]^2\} \int_{-\infty}^{+\infty} \exp(2bt - at^2)\,\mathrm{d}t \tag{B9}$$

where $\bar{\bar{\sigma}}$ appears only in the first factor, and $\bar{\sigma}$ appears in all other factors. The integration (27–30) applies equally well to (B8) and leads to

$$\bar{P} = [1/(2\bar{\bar{\sigma}}^2 a\sqrt{\pi})] \exp\{(b/a)^2 - [H/(2\bar{\sigma})]^2\} \tag{B10}$$

where (B8a) and (B8b) may be substituted:

$$\bar{P} = [\bar{\sigma}/(\bar{\bar{\sigma}}^2\sqrt{\pi})] \|\mathbf{V_1} - \mathbf{V_2}\|^{-1} \exp\{-[H/(2\bar{\sigma})]^2\} \times \{1 - [(\mathbf{V_1} - \mathbf{V_2}) \cdot \mathbf{e_z}]^2 \|\mathbf{V_1} - \mathbf{V_2}\|^{-2}\} \tag{B11}$$

This result simplifies to (32) when the rms position errors are equal $\sigma_1 = \sigma_2 = \sigma = \bar{\sigma} = \bar{\bar{\sigma}}$, and generalizes (32) when (B6a) and (B6b) they are unequal. For example, if $\sigma_1 = \sigma/2$, and $\sigma_2 = \sqrt{3}\sigma/2$, then $\overline{\sigma} = \sigma$ in (B6b) and $\overline{\overline{\sigma}} = \sqrt[4]{3}\sigma/2$ in (B6a), and (B11) coincides with (34) and (35) apart from a factor $\sigma\overline{\sigma}/\overline{\overline{\sigma}}^2 = 4/\sqrt{3}$.

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